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Using closed orbits to bifurcate many periodic solutions for pendulum-type equations

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Abstract

For pendulum-like equations, perturbation-type arguments and topological tools provide the existence of external forces with many associated periodic solutions.

1. Introduction and statement of the main results

In this work we deal with boundary value problems of the type

$$\begin{cases} u'' + cu' + g(u) = e(t) = \bar{e} + \tilde{e}(t), \\ u(T) - u(0) = k, \quad u'(T) - u'(0) = k'. \end{cases} \tag{1}$$

Here, k, k', T, c are given real constants with $T > 0$, $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ is a continuous, 2π-periodic function with zero mean ($\int_0^{2\pi} g(u)\,du = 0$), and $e \in L_1[0,T]$ is decomposed as $e = \bar{e} + \tilde{e}$, where

$$\bar{e} \in \mathbb{R}, \qquad \tilde{e} \in \tilde{L}_1[0,T] := \left\{ e \in L_1[0,T]\colon \frac{1}{T}\int_0^T e(s)\,ds = \frac{k' + ck}{T} \right\}.$$

Simply integrate both sides of the differential equation in (1) to check that, in case $g \equiv 0$, a necessary condition for the linear problem (1) to have a solution, is $e = \tilde{e} \in \tilde{L}_1[0,T]$, which is easily shown to be also sufficient. Further, the whole set of solutions can be

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obtained by adding all constant functions to any particular solution. This case being completely understood, we will always assume that g is nontrivial in what follows. On the other hand, the simple change of variables $\hat{u}(t) := u(T-t)$, $0 \leqslant t \leqslant T$, shows that it is not restrictive to assume $c \geqslant 0$.

Observe also that, in case u is a solution of (1), $u + 2\pi$ is again a solution. These solutions are called *geometrically equal* (they coincide when seen in the circumference $\mathbb{R}/2\pi\mathbb{Z}$), and our objective in this work is, for given T, k, k', c, g, to find external forcing terms e such that (1) has at least, or exactly, a prefixed even number $2n$ of *geometrically different* solutions.

This problem, which contains in particular the *periodic problem* ($k = k' = 0$) for the dissipative *pendulum equation* ($g(u) = \Lambda \sin(u)$), has therefore a long history that may be found, for instance, in [6]. As a consequence, many aspects of this problem are known even though also many important and profound questions remain still open.

Most results known for this problem deal with the periodic setting

$$\begin{cases} u'' + cu' + g(u) = e(t) = \bar{e} + \tilde{e}(t), \\ u(T) - u(0) = 0, \quad u'(T) - u'(0) = 0. \end{cases} \tag{2}$$

In this framework, it was proved in the pioneering work of Mawhin and Willem [7] that, if the problem is conservative ($c = 0$), for any given $e = \tilde{e} \in \tilde{L}_1[0,T] = \{h \in L_1[0,T]\colon \int_0^T h(s)\,ds = 0\}$, problem (2) has, at least, two different solutions. This result, which turns out to be false for the nonconservative case (just remember the first counterexample, given by Ortega [8], showing that, if $c \neq 0$, (2) may not have solutions at all even for $e = \tilde{e} \in \tilde{L}_1[0,T]$), was attained through the use of variational arguments.

More recently, it was proved by Donati [4] that, in the periodic problem for the conservative, forced pendulum equation ($g(u) = \Lambda \sin(u)$), it is always possible to find forcing terms $e = \tilde{e} \in \tilde{L}_1[0,T]$ such that (2) has, at least, four geometrically different solutions. This result was extended by Ortega [9], who established that, in the same framework, it is possible to change 4 by any number.

In this work we first show how Ortega's theorem keeps valid under the presence of a friction and for a bigger class of oscillating functions g, namely, those which are restriction to the real line of an entire function.

Theorem 1.1. *Assume there exists an entire function $\hat{g}: \mathbb{C} \to \mathbb{C}$ whose restriction to the real line is g. Then, for any $n \in \mathbb{N}$, the set*

$$\mathcal{S}_n := \big\{ e \in L_1[0,T] \text{ such that problem (1) has at least } n \text{ geometrically different solutions} \big\}$$

has nonempty interior in $L_1[0,T]$. Indeed,

(1) $\mathring{\mathcal{S}}_n \cap \tilde{L}_1([0,T]) \neq \emptyset$ *if $c = 0$,*
(2) $\mathring{\mathcal{S}}_n \cap \{e = \tilde{e} + \bar{e} \in L_1([0,T])\colon -\epsilon < \bar{e} < 0\} \neq \emptyset \neq \mathcal{S}_n \cap \{e = \bar{e} + \tilde{e} \in L_1([0,T])\colon 0 < \bar{e} < \epsilon\}$ *for every $\epsilon > 0$ if $c \neq 0$.*

As a consequence, in the framework of Theorem 1.1, there are analytic functions $e \in C^\omega([0,T])$ such that problem (1) has at least n solutions. On the other hand, in view of

Theorem 1.1, the following question arises: Is it true that $\mathring{\mathcal{S}}_n \cap \tilde{L}_1[0, T] \neq \emptyset$ independently of g, c? We do not answer to this question, which seems likely to be true.

Subsequently, we turn ourselves to the study of conservative, pendulum-type systems

$$\begin{cases} u'' + g(u) = e(t) = \bar{e} + \tilde{e}(t), \\ u(T) - u(0) = k, \quad u'(T) - u'(0) = k'. \end{cases} \tag{3}$$

This time we may use our better knowledge of the problem to explore *exact multiplicity* results. To get a feeling of what we should expect, observe that, in case g is $\frac{2\pi}{p}$-periodic for some $p \in \mathbb{N}$, the number of geometrically different solutions of (3) (or (1)), if finite, is always a multiple of p. Consequently, we impose a new assumption on g implying, in particular, that its minimal period is 2π.

(H) $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$ has a primitive G which attains its maximum only once in $[0, 2\pi[$.

Then, if the time period T is big enough, we may prove the existence of forcing terms $e = \tilde{e} \in \tilde{L}_1[0, 1]$ such that problem (3) has exactly a prefixed even number $2n$ of solutions.

Theorem 1.2. *Assume* (H). *Then, for each given $n \in \mathbb{N}$ there exists $T_0 = T_0(n) > 0$ such that, for any $T > T_0(n)$, there exists an open set $\mathcal{O}_{n,T} \subset L_1([0, T])$ with $\mathcal{O}_{n,T} \cap \tilde{L}_1[0, T] \neq \emptyset$, and with the property that for any $e \in \mathcal{O}_{n,T}$, problem* (3) *has exactly $2n$ geometrically different solutions.*

In particular cases, say, in the case of the pendulum equation, we are able to estimate the quantity $T_0(n)$. We obtain

Theorem 1.3. *Assume $g(x) = \Lambda \sin(x)$, $\Lambda \neq 0$, and let $n \in \mathbb{N}$ be given. If*

$$T \geqslant 12 \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \frac{n}{\sqrt{|\Lambda|}}$$

then, there exists an open set $\mathcal{O}_{n,T} \subset L_1([0, T])$ with $\mathcal{O}_{n,T} \cap \tilde{L}_1[0, T] \neq \emptyset$, such that for any $e \in \mathcal{O}_{n,T}$, problem (3) *has exactly $2n$ geometrically different solutions.*

Next result will follow from Theorem 1.2 above.

Corollary 1.4. *Assume* (H). *Then, for each given $n \in \mathbb{N}$, there exists a discrete and closed set $F_n \subset \mathbb{R}_+$, such that, for any $T \in \mathbb{R}_+ \setminus F_n$, there exists an open set $\mathcal{O}_{n,T} \subset L_1([0, T])$ with $\mathcal{O}_{n,T} \cap \tilde{L}_1[0, T] \neq \emptyset$, and with the property that, for any $e \in \mathcal{O}_{n,T}$, problem* (3) *has exactly $2n$ geometrically different solutions. In particular, there exists a countable subset F of $\mathbb{R}_+$ such that, for any $T \in \mathbb{R}_+ \setminus F$ and for any $n \in \mathbb{N}$, there exists an open set $\mathcal{O}_{n,T} \subset L_1([0, T])$ with $\mathcal{O}_{n,T} \cap \tilde{L}_1[0, T] \neq \emptyset$, with the property that, for any $e \in \mathcal{O}_{n,T}$, problem* (3) *has exactly $2n$ geometrically different solutions.*

Some remarks on the notation. Through this paper, a function of several variables, $S = S(x_1, x_2, \ldots, x_p)$, defined on an open subset of the Cartesian product of the Banach spaces $X_1, X_2, \ldots, X_p$, will be called C^1 (or continuously differentiable) with respect to x_i if it is continuous and the partial derivative $\partial_{x_i} S$ is continuously defined on the whole domain of S. We write $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ (so that $L_1(\mathbb{T}) \equiv L_1([0,T])$, $C(\mathbb{T}) \equiv \{f \in C([0,T])\colon\ f(T) - f(0) = 0\}$, $C^1(\mathbb{T}) \equiv \{f \in C^1([0,T])\colon\ f(T) - f(0) = 0 = f'(T) - f'(0)\}$, $W_{1,1}(\mathbb{T}) \equiv \{f \in W_{1,1}([0,T])\colon\ f(0) = f(T)\}$, etc.). Given $s \in \mathbb{R}$ we call τ_s the associated translation operator (defined by $\tau_s f(x) := f(s + x)$). A (real) trigonometric polynomial of degree $r \in \mathbb{N}$ on $\mathbb{T}$ is a function $P : \mathbb{T} \to \mathbb{R}$ of the form

$$P(t) = p_0 + \sum_{j=1}^{r} \left[p_j \cos\left(j \frac{2\pi}{T} t \right) + q_j \sin\left(j \frac{2\pi}{T} t \right) \right]$$

for some real coefficients p_j, q_j with $p_r^2 + q_r^2 \neq 0$, or, in complex notation,

$$P(t) = \sum_{j=-r}^{r} \omega_j e^{ij \frac{2\pi}{T} t}$$

for some complex coefficients ω_j with $\omega_{-j} = \bar{\omega}_j$ and $\omega_r \neq 0$.

2. The abstract framework: a bifurcation result

The implicit function theorem may be used to obtain the existence of nontrivial branches of solutions bifurcating from a trivial one. There are many results of this type in the literature, see, for instance, [2,3]. This section is devoted to recall some general bifurcation arguments, which we will need later.

Let X, Y be real Banach spaces, let $U \subset X$, $V \subset Y$ be open and $y_0 \in V$; let $I \subset \mathbb{R}$ be an open interval with $0 \in I$; finally, let $\mathcal{H} : I \times U \times V \to X$, $(\lambda, x, y) \mapsto \mathcal{H}(\lambda, x, y)$ be a C^1 mapping. We think of λ, x, y as being the bifurcation parameter, the variable, and an extra perturbation parameter, respectively.

We are interested in the solutions of the equation

$$\mathcal{H}(\lambda, x, y) = 0, \quad \lambda \in I,\ x \in U,\ y \in V, \tag{4}$$

for $\lambda \neq 0$.

We assume that for $(\lambda, y) = (0, y_0)$ there exists a trivial branch of solutions given by the C^1 curve $\gamma : \mathbb{R} \to U \subset X$,

$$\mathcal{H}\bigl(0, \gamma(s), y_0\bigr) = 0 \quad \forall s \in \mathbb{R}.$$

The curve γ is further assumed to have the following property. There exists some closed, linear hyperplane $\tilde{X} \subset X$ such that

$$\gamma'(s) \notin \tilde{X} \quad \forall s \in \mathbb{R} \tag{5}$$

(in particular, γ should be injective and $\gamma'(s) \neq 0$ $\forall s \in \mathbb{R}$). Deriving the equality above with respect to s, we obtain

$$\partial_x \mathcal{H}\bigl(0, \gamma(s), y_0\bigr)\gamma'(s) = 0 \quad \forall s \in \mathbb{R}, \tag{6}$$

and consequently,

$$0 \neq \gamma'(s) \in \ker \partial_x \mathcal{H}\bigl(0, \gamma(s), y_0\bigr) \quad \forall s \in \mathbb{R}.$$

We further assume that

(a) $\partial_x\mathcal{H}(0,\gamma(s),y_0): X\to X$ is a Fredholm operator of zero index for every $s\in\mathbb{R}$,
(b) $\dim\ker\partial_x\mathcal{H}(0,\gamma(s),y_0)=1$ (that is, $\ker\partial_x\mathcal{H}(0,\gamma(s),y_0)=\langle\gamma'(s)\rangle$) $\forall s\in\mathbb{R}$.

Hypothesis (a), (b) together with the equality

$$\bigl[\operatorname{im}\partial_x\mathcal{H}\bigl(0,\gamma(s),y_0\bigr)\bigr]^{\perp}=\ker\partial_x\mathcal{H}\bigl(0,\gamma(s),y_0\bigr)^{*}\quad\forall s\in\mathbb{R}\tag{7}$$

(the star denoting adjoint operator), imply that $\dim\ker\partial_x\mathcal{H}(0,\gamma(s),y_0)^*=1$, which allows us to use the implicit function theorem to obtain the existence of a continuous curve[1] $\sigma:\mathbb{R}\to X^*$ such that

$$\bigl\|\sigma(s)\bigr\|_*=1,\quad\bigl\langle\sigma(s)\bigr\rangle=\bigl[\operatorname{im}\partial_x\mathcal{H}\bigl(0,\gamma(s),y_0\bigr)\bigr]^{\perp}\quad\forall s\in\mathbb{R}.\tag{8}$$

Using a partition of the unity argument, it is not difficult to show now the existence of a C^∞ curve $m:\mathbb{R}\to X$ such that $\langle m(s),\sigma(s)\rangle\neq 0$, or, what is the same,

$$m(s)\notin\operatorname{im}\partial_x\mathcal{H}\bigl(0,\gamma(s),y_0\bigr)\quad\forall s\in\mathbb{R}.$$

Let us fix instants $-\infty<a<b<+\infty$ and denote $\mathcal{J}:=\,]a,b[$. For any $s\in\mathbb{R}$, the space X splits as $X=\langle m(s)\rangle\oplus\operatorname{im}\partial_x\mathcal{H}(0,\gamma(s),y_0)$ and also as $X=\langle\gamma'(s)\rangle\oplus X$. We use this latter splitting together with the inverse function theorem to uniquely write each element $x\in X$ in a small ('tubular') open neighborhood of $\gamma(\mathcal{J})$ as $x=\gamma(s)+\tilde{x}$, where $s\in\mathcal{J}$ and $\tilde{x}\in\tilde{X}$ is near 0, and we call $\Pi_s: X\to\langle m(s)\rangle\equiv\mathbb{R}$ the linear projection associated with the first one. Observe that

$$\Pi_s(x)=\frac{\langle x,\sigma(s)\rangle}{\langle m(s),\sigma(s)\rangle}m(s)\quad\forall s\in\mathbb{R},\ \forall x\in X.$$

With this notation, Eq. (4) can be rewritten as the system

$$(I_X-\Pi_s)\mathcal{H}\bigl(\lambda,\gamma(s)+\tilde{x},y\bigr)=0,\tag{9}$$

$$\Pi_s\mathcal{H}\bigl(\lambda,\gamma(s)+\tilde{x},y\bigr)=0.\tag{10}$$

This is the so-called *Lyapunov–Schmidt* system for (4). Usually, (9) is referred to as the auxiliary equation and (10) as the bifurcation equation of the system.

Our task will be to study the bifurcation branches, alongside with λ, of solutions of Eq. (4) emanating from the curve $\gamma|\mathcal{J}:\mathcal{J}\to X$. Using the implicit function theorem we may solve Eq. (9) near $\{0\}\times\gamma(\mathcal{J})\times\{y_0\}$, obtaining

Lemma 2.1. *There exist open sets $\mathcal{U}\subset X$ with $\gamma(\mathcal{J})\subset\mathcal{U}\subset U\cap(\gamma(\mathcal{J})+\tilde{X})$, $\mathcal{I}\subset\mathbb{R}$ with $0\in\mathcal{I}\subset I$, $\mathcal{V}\subset Y$ with $y_0\in\mathcal{V}\subset V$, and a continuous mapping $\Psi:\mathcal{I}\times\mathcal{J}\times\mathcal{V}\to\tilde{X}$ such that*

$$\bigl\{\tilde{x}\in\tilde{X}:\ \gamma(s)+\tilde{x}\in\mathcal{U},\ (I_X-\Pi_s)\mathcal{H}\bigl(\lambda,\gamma(s)+\tilde{x},y\bigr)=0\bigr\}=\bigl\{\Psi(\lambda,s,y)\bigr\},$$

[1] Observe that, due to hypothesis (b)—after, possibly, a reparametrization—γ will be of class $C^{p+1}(\mathbb{R})$, $p\geqslant 1$, provided $U\to X$, $x\mapsto\mathcal{H}(0,x,y_0)$ has class C^{p+1}. In this case, σ will be of class $C^p(\mathbb{R})$.

for any $\lambda \in \mathcal{I}$, $s \in \mathcal{J}$ and $y \in \mathcal{V}$. This means that, on $\mathcal{I} \times \mathcal{U} \times \mathcal{V}$, Eq. (4) reads as

$$\big\langle \mathcal{H}\big(\lambda, \gamma(s) + \Psi(\lambda, s, y), y\big), \sigma(s)\big\rangle = 0, \quad (\lambda, s, y) \in \mathcal{I} \times \mathcal{J} \times \mathcal{V}.$$

We start by exploring the structure of the solution set of this equation for $y = y_0$. We define

$$\xi : \mathcal{I} \times \mathcal{J} \subset \mathbb{R}^2 \to \mathbb{R}, \quad (\lambda, s) \mapsto \big\langle \mathcal{H}\big(\lambda, \gamma(s) + \Psi(\lambda, s, y_0), y_0\big), \sigma(s)\big\rangle.$$

Of course, ξ is a C^1 mapping with respect to λ and verifies $\xi(0, s) = 0$ $\forall s \in \mathcal{J}$. Further,

$$\begin{aligned} \partial_\lambda \xi(0, s) &= \big\langle \partial_\lambda \mathcal{H}\big(0, \gamma(s), y_0\big) + \partial_x \mathcal{H}\big(0, \gamma(s), y_0\big)\partial_\lambda \Psi(0, s, y_0), \sigma(s)\big\rangle \\ &= \big\langle \partial_\lambda \mathcal{H}\big(0, \gamma(s), y_0\big), \sigma(s)\big\rangle \quad \forall s \in \mathcal{J}. \end{aligned}$$

Therefore, the mapping $\vartheta : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ defined by the rule

$$(\lambda, s) \mapsto \begin{cases} \frac{1}{\lambda}\xi(\lambda, s) = \frac{1}{\lambda}\langle \mathcal{H}(\lambda, \gamma(s) + \Psi(\lambda, s, y_0), y_0), \sigma(s)\rangle & \text{if } \lambda \neq 0, \\ \langle \partial_\lambda \mathcal{H}(0, \gamma(s), y_0), \sigma(s)\rangle & \text{if } \lambda = 0, \end{cases}$$

is continuous. We recall that Eq. (4) with $y = y_0$, $\lambda \in \mathcal{I} \setminus \{0\}$, $x \in \mathcal{U}$, reduces to $\vartheta(\lambda, s) = 0$, $s \in \mathcal{J}$.

Thus, we are lead to consider the real-valued, continuous curve

$$\Gamma : \mathbb{R} \to \mathbb{R}, \quad s \mapsto \big\langle \partial_\lambda \mathcal{H}\big(0, \gamma(s), y_0\big), \sigma(s)\big\rangle. \tag{11}$$

A remarkable fact this formula is that no explicit mention to Ψ appears in the right-hand side, even though it was built using this function. In particular, *the curve $\Gamma : \mathcal{J} \to \mathbb{R}$ does not depend on the particular choices of $\tilde{X}, m$.*

It does not seem strange now that, under suitable nondegeneracy hypothesis, zeroes of Γ could be bifurcated to zeroes of $\xi(\lambda, \cdot)$ and, consequently, to zeroes of $\mathcal{H}(\lambda, \cdot, y_0)$ for $|\lambda|$ small. This is shown below.

Lemma 2.2. *Let $\mathcal{U}_*$ be any open subset of U with $\mathcal{U}_* \supset \gamma(\mathcal{J})$, and let $a < c_0 < c_1 < \cdots < c_p < b$ verify*

$$(-1)^i \Gamma(c_i) > 0, \quad i = 0, \ldots, p. \tag{12}$$

Then, there exists some $\epsilon_* > 0$ with $\mathcal{I}_* :=]0, \epsilon_*[\subset I$ such that $\mathcal{H}(\lambda, \gamma(c_i) + \Psi(\lambda, c_i, y_0), y_0) \in (-1)^i \mathbb{R}^+ m(c_i)$ $\forall \lambda \in \mathcal{I}_*$, $i = 0, \ldots, p$. In particular, for any $\lambda \in \mathcal{I}_*$, Eq. (4) *with $y = y_0$ has, at least, p different solutions $x \in \mathcal{U}$ for all $\lambda \in \mathcal{I}_*$. Furthermore, for any $\tilde{\lambda} \in \mathcal{I}_*$, there exist an open interval $\tilde{\mathcal{I}} \subset \mathcal{I}_*$ with $\tilde{\lambda} \in \tilde{\mathcal{I}}$ and an open set $\tilde{\mathcal{V}} \subset V$ with $y_0 \in \tilde{\mathcal{V}}$ such that $\mathcal{H}(\lambda, \gamma(c_i) + \Psi(\lambda, c_i, y), y) \in (-1)^i \mathbb{R}^+ m(c_i)$ $\forall \lambda \in \mathcal{I}_*$, $\forall y \in \tilde{\mathcal{V}}$, $i = 0, \ldots, p$. In particular, Eq.* (4) *has at least p different solutions $x \in \mathcal{U}_*$ for all $\lambda \in \tilde{\mathcal{I}}$, $y \in \tilde{\mathcal{V}}$.*

Of course, all this is a simple consequence of the continuity of ϑ; if it is positive somewhere, it remains positive in a neighborhood, and, whenever $\vartheta(\lambda, \cdot)$ has different sign at two instants c_i, c_{i+1}, it vanishes somewhere between them.

To proceed, we will need some extra regularity on $\mathcal{H}$. Namely, let us assume that both mappings

$$I \times U \to X, \quad (\lambda, x) \mapsto \partial_\lambda \mathcal{H}(\lambda, x, y_0)$$

and

$$I \times U \to L(X), \quad (\lambda, x) \mapsto \partial_x \mathcal{H}(\lambda, x, y_0)$$

are C^1 with respect to x. If this is the case, σ is a C^1 curve and ϑ is itself continuously differentiable with respect to t. In particular, $\Gamma : \mathbb{R} \to \mathbb{R}$ is C^1.

Let us call $U_{a,b}$ the open subset of U delimited by the (affine) hyperplanes $\gamma(a) + \tilde{X}$ and $\gamma(b) + \tilde{X}$. We further assume

(c) $\partial_y \mathcal{H} : I \times U_{a,b} \times V \to L(Y, X)$ and $\partial_\lambda \mathcal{H} : I \times U_{a,b} \times V \to X$ are bounded,
(d) for any sequence $\{x_n\}_n \subset U_{a,b}$ such that $\{\mathcal{H}(0, x_n, y_0)\} \to 0$, $\{\operatorname{dist}(x_n, \gamma(\mathbb{R}))\} \to 0$.

The purpose of these two hypothesis is to guarantee that given any open subset O of X containing $\gamma([a, b])$ there exist open sets $\mathcal{I}_* \subset I$ and $\mathcal{V}_*$ containing 0 and y_0, respectively, such that Eq. (4) has no solutions $x \in U_{a,b} \setminus O$ for any $(\lambda, y) \in \mathcal{I}_* \times \mathcal{V}_*$. In this way, under hypothesis ensuring the nondegeneracy of the zeroes of Γ, the implicit function theorem may be used to obtain precise results on the number of solutions of (4) for $x \in U_{a,b}$.

Lemma 2.3. *Assume $a < c_0 < c_1 < \cdots < c_p < b$ verify*

$$\Gamma(c_i) = 0, \quad \Gamma'(c_i) \neq 0, \quad i = 0, \ldots, p,$$
$$\Gamma(t) \neq 0 \quad \forall t \in [a, b] \setminus \{c_0, c_1, \ldots, c_p\}.$$

Then, there exist $\epsilon_* > 0$ with $]-\epsilon_*, \epsilon_*[:= \mathcal{I}_* \subset I$, and continuous curves $\gamma_1, \ldots, \gamma_p : \mathcal{I}_* \to U_{a,b} \subset X$ which are, further, C^1 on $\mathcal{I}_* \setminus \{0\}$, such that $\gamma_i(0) = \gamma(c_i)$, $1 \leqslant i \leqslant p$, and

$$\bigl\{(\lambda, x) \in \mathcal{I}_* \times U_{a,b}\colon \lambda \neq 0,\ \mathcal{H}(\lambda, x, y_0) = 0\bigr\} = \bigcup_{i=1}^{p} \bigl\{\bigl(\lambda, \gamma_i(\lambda)\bigr)\colon \lambda \in \mathcal{I}_*,\ \lambda \neq 0\bigr\}.$$

Moreover, given any $\tilde{\lambda} \in \mathcal{I}_*$, $\tilde{\lambda} \neq 0$, there exist an open interval $\tilde{\mathcal{I}} \subset I$ containing $\tilde{\lambda}$, an open subset $\tilde{\mathcal{V}} \subset Y$ with $y_0 \in \tilde{\mathcal{V}} \subset V$, and C^1 mappings $\tilde{s}_1, \ldots, \tilde{s}_p : \tilde{\mathcal{I}} \times \tilde{\mathcal{V}} \to U \subset X$ such that

$$\tilde{s}_i(\lambda, y_0) = \gamma_i(\lambda) \quad \forall \lambda \in \tilde{\mathcal{I}},$$
$$\bigl\{\bigl(\lambda, x, y\bigr) \in \tilde{\mathcal{I}} \times U_{a,b} \times \tilde{\mathcal{V}}\colon \mathcal{H}(\lambda, x, y) = 0\bigr\} = \bigcup_{i=1}^{p} \bigl\{\bigl(\lambda, \tilde{s}_i(\lambda, y), y\bigr)\colon (\lambda, y) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{V}}\bigr\}.$$

3. A functional framework for the periodic pendulum

The goal of this section is to establish the needed functional setting in order to reformulate problem (1) as a fixed point one for a regular mapping on a Banach space and apply the results in the last section.

Denoting by φ the only solution to the linear problem

$$\begin{cases} \varphi'' + c\varphi' = \tilde{e}(t), \\ \varphi(0) = 0, \quad \varphi(T) = k, \quad \varphi'(T) - \varphi'(0) = k', \end{cases}$$

the standard change of variables $v = u - \varphi$ transforms problem (1) into the periodic problem

$$\begin{cases} v'' + cv' + g(v + \varphi(t)) = \bar{e}, \\ v(T) - v(0) = 0, \quad v'(T) - v'(0) = 0. \end{cases} \tag{13}$$

It will be more convenient to work directly on this problem rather than with the original one. Namely, for any and given $\varphi \in L_1(\mathbb{T})$ and $\bar{e} \in \mathbb{R}$ we may consider the problem

$$v'' + cv' + g\big(v + \varphi(t)\big) = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}). \tag{14}$$

We define the linear differential operator

$$\mathcal{L}_0 : W_{2,1}(\mathbb{T}) \to L_1(\mathbb{T}), \quad \mathcal{L}_0(v) := v'' \ \forall v \in W_{2,1}(\mathbb{T}),$$

and the Nemytskii operator associated with g,

$$\mathcal{N}_g : L_1(\mathbb{T}) \to L_1(\mathbb{T}),$$
$$\big[\mathcal{N}_g(v)\big](x) := g\big(v(x)\big) \quad \forall x \in \mathbb{T},\ \forall v \in L_1(\mathbb{T}),$$

so that (14) is equivalent to the functional equation

$$\mathcal{L}_0(v) + cv' + \mathcal{N}_g(v + \varphi) = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}). \tag{15}$$

The operator $\mathcal{L}_0$ is not injective, but (15) is not changed if the same quantity v is subtracted and added, to get the equality

$$\big[\mathcal{L}_0(v) - v\big] + \big[\mathcal{N}_g(v + \varphi) + v + cv'\big] = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}), \tag{16}$$

whose first term is invertible. We denote by $\mathcal{K}$ the inverse operator of $v \mapsto \mathcal{L}_0(v) - v$, which is a compact operator when seen from $L_1(\mathbb{T})$ to $W_{1,1}(\mathbb{T})$. We also observe that $\mathcal{K}$ is 'self-adjoint' in the sense that

$$\int_0^T \big[\mathcal{K}(f)\big](x) g(x)\, dx = \int_0^T f(x)\big[\mathcal{K}(g)\big](x)\, dx \quad \forall f, g \in \mathbb{L}_1(\mathbb{T}). \tag{17}$$

In this way, Eq. (16) can be rewritten as a fixed point problem

$$v = -\mathcal{K}\big[\mathcal{N}_g(v + \varphi) + v + cv' - \bar{e}\big] = -\mathcal{K}\big[\mathcal{N}_g(v + \varphi) + v + cv'\big] - \bar{e},$$
$$v \in W_{1,1}(\mathbb{T}). \tag{18}$$

We fix ψ_0 in $W_{1,1}(\mathbb{T})$ (which will be determined later) and define

$$\mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times \big[\mathbb{R} \times L_1(\mathbb{T})\big] \to W_{1,1}(\mathbb{T}),$$
$$(\lambda, v, \bar{e}, \varphi) \mapsto v + \mathcal{K}\big[\mathcal{N}_g(v + \lambda\psi_0 + \varphi) + v + cv'\big] + \bar{e}. \tag{19}$$

It is easily checked that $\mathcal{H}$ is C^1 and the continuous, linear operator $\partial_v\mathcal{H}(\lambda, v, \bar{e}, \varphi)$: $W_{1,1}(\mathbb{T}) \to W_{1,1}(\mathbb{T})$ has the form identity minus compact for any $(\lambda, v, \bar{e}, \varphi)$, so that (a) is automatically satisfied. Furthermore, the partial derivatives $\partial_\lambda\mathcal{H}, \partial_{\bar{e}}\mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times [\mathbb{R} \times L_1(\mathbb{T})] \to W_{1,1}(\mathbb{T})$, and $\partial_\varphi\mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times [\mathbb{R} \times L_1(\mathbb{T})] \to L(L_1(\mathbb{T}), W_{1,1}(\mathbb{T}))$ are clearly bounded, as required in (c). Finally, it is easily checked that, in case $g \in C^2(\mathbb{R})$, both mappings

$$\mathbb{R} \times W_{1,1}(\mathbb{T}) \to W_{1,1}(\mathbb{T}), \quad (\lambda, v) \mapsto \partial_\lambda \mathcal{H}(\lambda, v, \bar{e}, \varphi),$$
$$\mathbb{R} \times W_{1,1}(\mathbb{T}) \to L\big(W_{1,1}(\mathbb{T})\big), \quad (\lambda, v) \mapsto \partial_v \mathcal{H}(\lambda, v, \bar{e}, \varphi),$$

are continuously differentiable with respect to v for any $(\bar{e}, \varphi) \in \mathbb{R} \times L_1(\mathbb{T})$.

In order to position ourselves in the abstract framework studied in previous section we still have to find $\bar{e} \in \mathbb{R}$ and $\varphi \in L_1(\mathbb{T})$ such that (18) has a whole nontrivial curve of solutions. Alternatively, we may try to find $e = \bar{e} + \tilde{e} \in L_1(\mathbb{T})$, $k, k' \in \mathbb{R}$ such that problem (1) has a curve of solutions.

The following proposition has interest by its own.

Proposition 3.1. *There exists a unique constant external force $\bar{e} = \bar{e}_{c,T} \in \mathbb{R}$ such that*

$$\begin{cases} u'' + cu' + g(u) = \bar{e}, \\ u(0) = 0, \quad u(t+T) = 2\pi + u(t) \quad \forall t \in \mathbb{R}, \end{cases} \tag{20}$$

has solution. This solution is unique (*we will call it $u_{c,T}$*) *and verifies*

$$u'_{c,T}(t) > 0 \quad \forall t \in \mathbb{R},$$
$$u_{c,T}\left(\frac{T}{2\pi} t\right) \underset{T\to 0}{\longrightarrow} t \quad \textit{uniformly w.r.t. } t \in \mathbb{R} \textit{ for } c > 0 \textit{ fixed.} \tag{21}$$

Finally,

$$\bar{e}_{0,T} = 0 \quad \forall T > 0, \qquad \bar{e}_{c,T} > \frac{2\pi}{T} c \quad \forall c, T > 0,$$
$$\bar{e}_{c,T} - \frac{2\pi}{T} c \underset{T\to 0}{\longrightarrow} 0 \quad \textit{for } c > 0 \textit{ fixed.} \tag{22}$$

Proof. Observe that condition $u(0) = 0$, which appears in (20), is nothing but a normalization condition. By this, we mean that, since our equation is autonomous and every solution to

$$\begin{cases} u'' + cu' + g(u) = \bar{e}, \\ u(t+T) = 2\pi + u(t) \quad \forall t \in \mathbb{R}, \end{cases} \tag{23}$$

verifies $\lim_{t\to+\infty} u(t) = +\infty$; $\lim_{t\to-\infty} u(t) = -\infty$, solutions to (23) are, up to translations in the time variable t, solutions to (20). Therefore, in order to find $\bar{e} \in \mathbb{R}$ such that (20) has at least one solution, it suffices to show the existence of $\bar{e} \in \mathbb{R}$ such that (23) has some solution. At this point we introduce the change of variables $v(t) := u(t) - \frac{2\pi}{T} t$, which transforms (23) into

$$\begin{cases} v'' + cv' + g\big(\frac{2\pi}{T} t + v\big) = \bar{e} - \frac{2\pi c}{T}, \\ v(t+T) = v(t) \quad \forall t \in \mathbb{R}, \end{cases} \tag{24}$$

and the existence of the constant $\bar{e}$ we were looking for, follows now from Schauder's fixed point theorem. Thus, we may fix such an $\bar{e}_{c,T} \in \mathbb{R}$ and a corresponding solution $u_{c,T}$ to (20) for $\bar{e} = \bar{e}_{c,T}$, $v_{c,T}(t) := u_{c,T}(t) - \frac{2\pi}{T} t$. Now, for $\bar{e} = \bar{e}_{c,T}$, $t \mapsto u_{c,T}(t+s)$ is a solution of (23) for every $s \in \mathbb{R}$ and, consequently, $t \mapsto u_{c,T}(t+s) - \frac{2\pi}{T} t = v_{c,T}(t+s) + \frac{2\pi}{T} s$ is a solution to (24) for every $s \in \mathbb{R}$.

Second order, periodic problems such as (24), having a nontrivial curve

$$\gamma : \mathbb{R} \to W_{1,1}(\mathbb{T}), \quad s \mapsto \tau_s v_{c,T} + \frac{2\pi}{T} s, \tag{25}$$

of solutions for some value $\bar{e}_{c,T}$ of $\bar{e}$ are usually called degenerate, and have been extensively studied in the literature. In particular, it is known (see [10] and references therein), that system (24) cannot have solutions for $\bar{e} \neq \bar{e}_{c,T}$ and not other solutions than $\{\gamma(s)\colon s \in \mathbb{R}\}$ for $\bar{e} = \bar{e}_{c,T}$. We shortly recall the argument for completeness. Let us take $\bar{e} \in \mathbb{R}$ such that (24) has a solution u. We consider the quantities

$$\alpha := \min\bigl\{s \in \mathbb{R}\colon \exists t \in \mathbb{R} \text{ with } u(t) = \bigl[\gamma(s)\bigr](t)\bigr\}, \tag{26}$$

$$\beta := \max\bigl\{s \in \mathbb{R}\colon \exists t \in \mathbb{R} \text{ with } u(t) = \bigl[\gamma(s)\bigr](t)\bigr\}. \tag{27}$$

Then, there exist $t_\alpha, t_\beta \in \mathbb{R}$ such that

$$\bigl[\gamma(\alpha)\bigr](t_\alpha) = u(t_\alpha), \qquad \bigl[\gamma(\alpha)\bigr]'(t_\alpha) = u'(t_\alpha), \qquad \bigl[\gamma(\alpha)\bigr](t) \leqslant u(t) \quad \forall t \in \mathbb{R},$$
$$\bigl[\gamma(\beta)\bigr](t_\beta) = u(t_\beta), \qquad \bigl[\gamma(\beta)\bigr]'(t_\beta) = u'(t_\beta), \qquad \bigl[\gamma(\beta)\bigr](t) \geqslant u(t) \quad \forall t \in \mathbb{R},$$

and we obtain

$$\begin{aligned}\bar{e}_{c,T} - \frac{2\pi}{T} c &= \bigl[\gamma(\alpha)\bigr]''(t_\alpha) + c\bigl[\gamma(\alpha)\bigr]'(t_\alpha) + g\biggl(\bigl[\gamma(\alpha)\bigr](t_\alpha) + \frac{2\pi}{T} t_\alpha\biggr) \\ &\leqslant u''(t_\alpha) + cu'(t_\alpha) + g\biggl(u(t_\alpha) + \frac{2\pi}{T} t_\alpha\biggr) = \bar{e} - \frac{2\pi}{T} c\end{aligned}$$

so that

$$\bar{e}_{c,T} \leqslant \bar{e}$$

and similarly, comparing u and $\gamma(\beta)$ in a neighborhood of t_β, we get

$$\bar{e}_{c,T} \geqslant \bar{e}$$

obtaining the equality $\bar{e} = \bar{e}_{c,T}$. Now,

$$\bigl[\gamma(\alpha)\bigr](t_\alpha) = u(t_\alpha), \qquad \bigl[\gamma(\alpha)\bigr]'(t_\alpha) = u'(t_\alpha)$$

so that $\gamma(\alpha) = u$. Similarly, $\gamma(\beta) = u$.

A similar reasoning shows indeed that the curves $\gamma(a)$ and $\gamma(b)$ do not intersect as soon as $a \neq b$. Otherwise, there would exist $a, b \in \mathbb{R}$ with $a < b$ and $\hat{t} \in \mathbb{R}$ such that $[\gamma(a)](\hat{t}) = [\gamma(b)](\hat{t})$. We may define $u := \gamma(b)$, and α as in (27), and the argument above shows that $\gamma(b) = u = \gamma(\alpha)$, which is a contradiction since $\alpha \leqslant a < b$ and consequently, $\gamma(\alpha)$ and $\gamma(b)$ have different mean. And we conclude that

$$a < b \to \bigl[\gamma(a)\bigr](t) < \bigl[\gamma(b)\bigr](t) \quad \forall t \in \mathbb{T}.$$

It means also that no different solutions to system (20) ($\bar{e} = \bar{e}_{c,T}$) intersect. On the other hand, as $u_{c,T}(t+T) = u_{c,T}(t) + 2\pi$, there exists some point $t_0 \in \mathbb{R}$ such that $u'_{c,T}(t_0) > 0$. Let us assume that the same inequality did not hold always and let t_1 be the minimum of those $t > t_1$ such that $u'_{c,T}(t) = 0$. Being $u_{c,T}$ a solution of the autonomous equation (20) ($\bar{e} = \bar{e}_{c,T}$), which is not an equilibrium, $u''_{c,T}(t_1) \neq 0$, and we deduce $u''_{c,T}(t_1) < 0$.

In this way, for $s \neq 0$ small, $u_{c,T}$ and $t \mapsto u_{c,T}(t+s)$ are different solutions to (20)—they are different at t_0—but intersecting near t_1, which is a contradiction.

Being g bounded, it follows from (24) that, for fixed $c > 0$,

$$\left\| v_{c,T}(\cdot) - \frac{1}{T}\int_0^T v_{c,T}(s)\,ds \right\|_{L_\infty[0,T]} \to 0 \quad \text{as } T \to 0,$$

so that, as stated, $u_c\left(\frac{T}{2\pi}t\right) \to t$ uniformly with respect to $t \in \mathbb{R}$ as $T \to 0$. Finally, to prove (22), just multiply Eq. (24) by $\frac{2\pi}{T} + v'$ and integrate on $[0, T]$, to get

$$\frac{c}{2\pi}\int_0^T v'_{c,T}(s)^2\,ds = \bar{e}_{c,T} - \frac{2\pi c}{T}$$

so that $\bar{e}_{0,T} = 0$, $\bar{e}_{c,T} > \frac{2\pi}{T}c$ $\forall c$, $T > 0$. Furthermore,

$$\begin{aligned}\bar{e}_{c,T} - \frac{2\pi}{T}c &= \frac{1}{T}\int_0^T g\left(\frac{2\pi}{T}s + v_{c,T}(s)\right)ds \\ &= \frac{1}{2\pi}\int_0^{2\pi} g\left(s + v_{c,T}\left(\frac{T}{2\pi}s\right)\right)ds \underset{T\to 0}{\longrightarrow} \frac{1}{2\pi}\int_0^{2\pi} g(s)\,ds = 0. \qquad \square\end{aligned}$$

Remark 3.2. Assume now $c \in \mathbb{R}$ is fixed. The mapping

$$\begin{aligned}\Psi : &\left\{ g \in C^1(\mathbb{R}/2\pi\mathbb{Z}) \setminus \{0\}\colon \int_0^T g(s)\,ds = 0 \right\} \\ &\to \left\{ v \in C^3(\mathbb{T})\colon\ v(0) = 0,\ \frac{2\pi}{T} + v'(t) > 0\ \forall t \in \mathbb{R} \right\}\end{aligned}$$

mapping g into the only solution v to (24) with $\bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c$ verifying $v(0) = 0$ is continuous. Furthermore, it is clearly bijective, its inverse being given by the rule

$$v \mapsto -\left(v'' + cv'\right) \circ \left[\frac{2\pi}{T}\iota + v\right]^{-1} + \frac{1}{T}\int_0^T \left(v'' + cv'\right) \circ \left[\frac{2\pi}{T}\iota + v\right]^{-1}(x)\,dx$$

($\iota(t) := t$ $\forall t \in \mathbb{R}$), which is also continuous. Then, both laws are homeomorphisms and it is easily checked that they conserve regularity

$$\Psi(g) \in C^{n+2}(\mathbb{T}) \quad \Leftrightarrow \quad g \in C^n(\mathbb{R}/2\pi\mathbb{Z}) \quad \forall n \geqslant 1.$$

In particular, for any trigonometric polynomial

$$P(t) = p_0 + \sum_{j=1}^{r}\left[p_j \cos\left(j\frac{2\pi}{T}t\right) + q_j \sin\left(j\frac{2\pi}{T}t\right)\right] \quad \text{with } P'(t) > -\frac{2\pi}{T}\ \forall t \in \mathbb{T},$$

there exists $g \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ whose associated curve $\Psi(g)$ is exactly P.

4. Many periodic solutions bifurcating from a closed loop at a constant external force

Thus, we have found that the equation

$$\mathcal{H}(\lambda, v, \bar{e}, \varphi) = 0, \quad v \in W_{1,1}(\mathbb{T}),$$

with $\mathcal{H}$ given in (19), has a nontrivial curve γ (given in (25)), of solutions for $\lambda = 0$,

$$\bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c, \qquad \varphi = \varphi_0(t) := \frac{2\pi}{T}t - \mathrm{Ent}\left(\frac{2\pi}{T}t\right).$$

To set ourselves under the framework of Section 2, we still have to check

(b) $\dim[\ker \partial_x \mathcal{H}(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0)] = 1$ (that is, $\langle \gamma'(s) \rangle = \ker(\partial_x \mathcal{H}(0, \gamma(s), \bar{e}_{c,T}, \varphi_0))$ for every $s \in \mathbb{R}$.

This is to say that the only solutions of the linear problem

$$\begin{cases} w'' + cw' + g'(u_{c,T}(t+s))w = 0, \\ w(t+T) = w(t) \quad \forall t \in \mathbb{R}, \end{cases} \tag{28}$$

should be the scalar multiples of $\tau_s u'_{c,T}$, for every $s \in \mathbb{R}$. Equivalently, the only T-periodic solutions of Hill's equation

$$w'' + cw' + g'\big(u_{c,T}(t)\big)w = 0 \tag{29}$$

should be the scalar multiples of $u'_{c,T}$. To see this we apply the reduction of order method; we already know that $u'_{c,T}$ is a solution to (29) and we conclude that

$$w_{c,T}(t) := u'_{c,T}(t) \int_0^t \frac{e^{-cr}\,dr}{u'_{c,T}(r)^2}$$

is another. Of course, this latter is not T-periodic,

$$w_{c,T}(0) = 0, \qquad w_{c,T}(T) > 0.$$

We next establish (d) for any $a < b \in \mathbb{R}$. With this aim, take any sequence $\{v_n\} \subset W_{1,1}(\mathbb{T})$ such that

$$\left\{\mathcal{H}\left(0, v_n, \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right)\right\}_n \to 0, \quad \left\{\frac{1}{T}\int_0^T v_n(t)\,dt\right\}_n \text{ bounded.}$$

For any $n \in \mathbb{N}$, write $v_n := \bar{v}_n + \tilde{v}_n$, $\bar{v}_n := \frac{1}{T}\int_0^T v_n(t)\,dt$, $\tilde{v}_n \in \tilde{X}$. By hypothesis, $\{\bar{v}_n\}$ is bounded, so that it has some convergent subsequence. Let us check that the same thing happens also for $\{\tilde{v}_n\}$. We call, for each $n \in \mathbb{N}$, $\theta_n := \mathcal{H}\big(0, v_n, \bar{e} - \frac{2\pi}{T}c, \varphi_0\big)$, so that

$$\tilde{v}_n + \mathcal{K}[\tilde{v}_n + c\tilde{v}'_n] = -\mathcal{K}\big[\mathcal{N}_g\big(\bar{v}_n + \tilde{v}_n + c\tilde{v}'_n + \varphi_0\big)\big] - \bar{e}_{c,T} + \frac{2\pi}{T}c + \theta_n \quad \forall n \in \mathbb{N}. \tag{30}$$

The sequence $\{\mathcal{N}_g(\bar{v}_n + \tilde{v}_n + c\tilde{v}_n' + \varphi_0)\}$ being bounded in $L_\infty(\mathbb{T})$, there exists a subsequence $\{v_{\sigma(n)}\}$ of $\{v_n\}$ such that $\{\mathcal{K}[\mathcal{N}_g(\bar{v}_{\sigma(n)} + \tilde{v}_{\sigma(n)} + c\tilde{v}'_{\sigma(n)} + \varphi_0)]\}_n$ is convergent in $W_{1,1}(\mathbb{T})$. As the operator $v \mapsto v + \mathcal{K}[v + cv']$ is a linear homeomorphism when seen from $\tilde{X}$ to its image (endowed with the $W_{1,1}(\mathbb{T})$ topology), we deduce from (30) that $\{\tilde{v}_{\sigma(n)}\}$ itself converges in $\tilde{X}$. Thus, there exists a convergent subsequence of $\{v_n\}$ and the limit must be a zero of $\mathcal{H}\big(0, \cdot, \bar{e} - \frac{2\pi}{T}c, \varphi_0\big)$. However, the set of zeroes of this mapping, as shown in Proposition 3.1, reduces to $\gamma(\mathbb{R})$, implying (d). We finally note that hypothesis (c) holds as soon as $g \in C^2(\mathbb{R})$.

To proceed further with the scheme of Section 2, let us pick a nonzero T-periodic solution $\nu_{c,T}$ of the adjoint equation of (29),

$$\omega'' - c\omega' + g'\big(u_{c,T}(t)\big)\omega = 0. \tag{31}$$

In the conservative case, problem (28) is selfadjoint and $\nu_{0,T}$ can be taken as $u'_{0,T}$. Consequently, $\nu_{0,T}$ does not change sign on $\mathbb{T}$. Let us see that the same thing happens for $\nu_{c,T}$ when $c \in \mathbb{R}$ is arbitrary.

Lemma 4.1. *For any $c \in \mathbb{R}$, consider the Hill's equation*

$$y'' + cy' + \alpha(t)y = 0, \tag{E_c}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a given locally integrable, T-periodic function. Then, (E_c) *has a T-periodic, positive solution if and only if* (E_{-c}) *has a T-periodic, positive solution.*

Proof. The solutions of (E_c) are related with those of (E_{-c}) by the rule

$$y(t) \text{ is a solution of } (\mathrm{E}_c) \quad \Leftrightarrow \quad z(t) = e^{ct}y(t) \text{ is a solution of } (\mathrm{E}_{-c}). \tag{32}$$

Using a Sturm–Liouville argument we know that, in case (E_c) has a never vanishing solution, the equation is disconjugate, meaning that any other nonzero solution of (E_c) vanishes, at most, at one single point in $\mathbb{R}$. Thanks to (32) we know that also (E_{-c}) is disconjugate, and therefore, its periodic solution cannot vanish. □

Observe that, for any $s \in \mathbb{R}$, $\tau_s \nu_{c,T}$ is a solution to the adjoint problem of (28),

$$\begin{cases} \omega'' - c\omega' + g'(u_{c,T}(t+s))\omega = 0, \\ \omega(t+T) = \omega(t) \quad \forall t \in \mathbb{R}. \end{cases} \tag{33}$$

Thus, given $h \in L_1(\mathbb{T})$, the nonhomogeneous, linear problem

$$\begin{cases} w'' + cw' + g'(u_{c,T}(t+s))w = h(t), \\ w(t+T) = w(t) \quad \forall t \in \mathbb{R}, \end{cases}$$

has solution if and only if $\int_0^T h(t)\nu_{c,T}(s+t)\,dt = 0$. Using (17) we deduce

$$\operatorname{im} \partial_x \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right)$$
$$= \left\{ v \in W_{1,1}(\mathbb{T}): \int_0^T v(t)\big[\mathcal{L}_0(\tau_s \nu_{c,T}) - \tau_s \nu_{c,T}\big](t)\,ds = 0 \right\}$$

so that, thanks to Lemma 4.1 above, we may take $m(s) \equiv 1\ \forall s \in \mathbb{R}$, and

$$\left[\operatorname{im} \partial_x \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right)\right]^{\perp} = \langle \mathcal{L}_0(\tau_s \nu_{c,T}) - \tau_s \nu_{c,T} \rangle$$
$$= \langle \tau_s(\mathcal{L}_0(\nu_{c,T}) - \nu_{c,T}) \rangle \quad \forall s \in \mathbb{R},$$

equality where the identifications $L_2(\mathbb{T}) \equiv L_2(\mathbb{T})^* \subset W_{1,1}(\mathbb{T})^*$ have been utilized. In this way, we obtain a explicit form for the curve σ in (8),

$$\sigma : \mathbb{R} \to L_2(\mathbb{T}) \subset W_{1,1}(\mathbb{T})^*, \quad s \mapsto \tau_s[\mathcal{L}_0(\nu_{c,T}) - \nu_{c,T}].$$

Finally, we are lead to consider the real valued, continuous curve $\Gamma : \mathbb{R} \to \mathbb{R}$ given by

$$s \mapsto \left\langle \partial_\lambda \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right), \sigma(s) \right\rangle$$
$$= -\int_0^T \left(\mathcal{K}[\mathcal{N}_{g'}(\gamma(s) + \varphi_0)\psi_0]\right)\left(\mathcal{L}_0(\tau_s \nu_{c,T}) - \tau_s \nu_{c,T}\right) dt$$
$$= -\int_0^T g'(u_{c,T}(t+s))\nu_{c,T}(t+s)\psi_0(t)\, dt$$
$$= \int_0^T \left(\nu''_{c,T}(t+s) - c\nu'_{c,T}(t+s)\right)\psi_0(t)\, dt, \tag{34}$$

that is, the convolution of $\nu''_{c,T} - c\nu'_{c,T}$ and ψ_0. In the conservative case, $\nu_{0,T} = u'_{0,T}$ and Γ is the convolution of $u'''_{0,T}$ and ψ_0.

The following result is now an straightforward consequence of Lemma 2.2.

Proposition 4.2. *Let*

$$A_n := \frac{2}{T}\int_0^T \nu_{c,T}(t)\cos\left(n\frac{2\pi}{T}t\right) dt, \qquad B_n := \frac{2}{T}\int_0^T \nu_{c,T}(t)\sin\left(n\frac{2\pi}{T}t\right) dt$$

be the sequences of Fourier coefficients of $\nu_{c,T}$. We assume that, for some $n_0 \in \mathbb{N}$,

$$A_{n_0}^2 + B_{n_0}^2 \neq 0.$$

Then, given any $\epsilon > 0$, it is possible to find $\varphi \in C^\infty(\mathbb{T})$ and $v_0, \ldots, v_{2n_0} \in C^2(\mathbb{T})$, $\varrho_0, \ldots, \varrho_{2n_0} \in \mathbb{R}_+$ such that

$$v_{2n_0}(t) = v_0(t) + 2\pi \quad \forall t \in \mathbb{T},\ \varrho_0 = \varrho_{2n_0}, \tag{35}$$

$$v''_q(t) + cv'_q(t) + g(v_q(t) + \varphi(t)) = \bar{e}_{c,T} - \frac{2\pi}{T}c + (-1)^q \varrho_q$$
$$\forall t \in \mathbb{T},\ \forall q = 0, \ldots, 2n_0, \tag{36}$$

$$v_{q-1}(t) < v_q(t) \quad \forall t \in \mathbb{T},\ \forall q = 1, \ldots, 2n_0. \tag{37}$$

Proof. Write $v_{c,T}$ as the sum of its Fourier series

$$v_{c,T}(t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi}{T}nt\right) + B_n \sin\left(\frac{2\pi}{T}nt\right)\right] + A_0.$$

Being $v_{c,T} \in C^2(\mathbb{T})$, we are allowed to derivate twice in the infinite sum above to get

$$v''_{c,T}(t) + cv'_{c,T}(t) = \sum_{n=1}^{\infty}\left[\left[-n^2\left(\frac{2\pi}{T}\right)^2 A_n + cn\frac{2\pi}{T}B_n\right]\cos\left(\frac{2\pi}{T}nt\right) + \left[-cn\frac{2\pi}{T}A_n - n^2\left(\frac{2\pi}{T}\right)^2 B_n\right]\sin\left(\frac{2\pi}{T}nt\right)\right].$$

Observe now that, if for some $n \in \mathbb{N}$,

$$-n^2\left(\frac{2\pi}{T}\right)^2 A_n + cn\frac{2\pi}{T}B_n = 0,$$

$$-cn\frac{2\pi}{T}A_n - n^2\left(\frac{2\pi}{T}\right)^2 B_n = 0,$$

then, $A_n = 0 = B_n$, since the determinant of the linear system is strictly positive. We conclude that

$$C_{n_0} := \left[cn_0\frac{2\pi}{T}B_{n_0} - n_0^2\left(\frac{2\pi}{T}\right)^2 A_{n_0}\right]^2 + \left[-cn_0\frac{2\pi}{T}A_{n_0} - n_0^2\left(\frac{2\pi}{T}\right)^2 B_{n_0}\right]^2 > 0.$$

At this point, we choose $\psi_0(t) = \frac{2}{T}\cos\left(\frac{2\pi}{T}n_0 t\right)$ in (34). We obtain

$$\Gamma(s) = \tilde{A}_{n_0}\cos\left(n_0\frac{2\pi}{T}s\right) + \tilde{B}_{n_0}\sin\left(n_0\frac{2\pi}{T}s\right)$$

for some $\tilde{A}_{n_0}, \tilde{B}_{n_0} \in \mathbb{R}$ with $\sqrt{\tilde{A}^2_{n_0} + \tilde{B}^2_{n_0}} = \sqrt{C_{n_0}} > 0$. This function has exactly $2n_0$ zeroes in $[0, T[$ and, on each one, its derivative does not vanish. The theorem follows from Lemma 2.2 $\square$

From such a scheme of ordered lower-upper-lower-upper... solutions, it follows immediately the existence of at least n_0 (geometrically) different solutions for the equation $\left\{v'' + cv' + g(v + \varphi(t)) = \bar{e}_{c,T} - \frac{2\pi}{T}c\right\}$—one between each pair of consecutive ordered lower and upper solutions. The three solutions theorem (see [1]) in fact implies the existence of *at least* $2n_0$ different solutions for this same equation. These solutions turn to come from mappings with nonzero degree so that all this keeps its validity under small perturbations. We state the precise result below

Proposition 4.3. *Let* $f_0 : [0, T] \times \mathbb{R}$, $(t, x) \mapsto f_0(t, x)$ *be continuous. Assume* $v_0, v_1, v_2, v_3 \in C^2(\mathbb{T})$ *verify*

(1) $v_0(t) < v_1(t) < v_2(t) < v_3(t)\ \forall t \in \mathbb{R}$,
(2) $(-1)^i[v''_i(t) + cv'_i(t) + f_0(t, v_i(t))] > 0\ \forall t \in \mathbb{T}$, $i = 0, \ldots, 3$.

Then, there exists $\epsilon > 0$ *such that, for any Carathéodory function* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *with*

$$\int_0^T \sup_{x\in\mathbb{R}} \left(\left\{\left|f_0(t,x) - f(t,x)\right|\right\}\right) dt < \epsilon, \tag{38}$$

the perturbed problem

$$\begin{cases} w'' + cw' + f(t,w) = 0, \\ w \in W_{2,1}(\mathbb{T}), \end{cases} \tag{39}$$

has at least three solutions w_1, w_2, w_3 *verifying*

(1) $v_0(t) < w_1(t) < v_1(t)$, $v_2(t) < w_3(t) < v_3(t)$ $\forall t \in \mathbb{T}$,
(2) $v_1(\hat{t}) < w_2(\hat{t}) < v_2(\hat{t})$ *for some* $\hat{t} \in \mathbb{T}$.

Along next results, it will be necessary to take into account, not only the time period T, which was, so far, fixed, but also all its divisors. Let us call, for any $m \in \mathbb{N}$, $A_{n,m}$ and $B_{n,m}$ the respective quantities A_n and B_n corresponding to the time period $\frac{T}{m}$.

Corollary 4.4. *Assume that, for some* $n, m \in \mathbb{N}$, $A_{n,m}^2 + B_{n,m}^2 \neq 0$. *Then, there exists an open set* $\mathcal{O} \subset L_1(\mathbb{T})$ *with* $\mathcal{O} \cap \left\{e = \tilde{e} + \bar{e} \in L_1(\mathbb{T}):\ \bar{e} = \bar{e}_{c,T/m} - \frac{2\pi}{T} mc\right\} \neq \emptyset$ *such that, for any* $e \in \mathcal{O}$, *problem* (1) *has at least* n *geometrically different solutions.*

Proof. From Proposition 4.2 we know the existence of $\varphi \in C^\infty\left(\mathbb{R}/\frac{T}{m}\mathbb{Z}\right)$ and a scheme of lower and upper solutions as given there on the interval $\left[0, \frac{T}{m}\right]$. These give rise to a corresponding scheme of ordered lower and upper solutions associated to $\varphi \in C^\infty(\mathbb{T})$ on the interval $[0, T]$. The result follows now from Proposition 4.3. □

Corollary 4.5. *Let* $n \in \mathbb{N}$ *be given, and assume that, for infinitely many* $m \in \mathbb{N}$, $\nu_{c,T/m}$ *is not a trigonometric polynomial of degree strictly lower than* n. *Then, for any* $\epsilon > 0$, *there exists an open set* $\mathcal{O} = \mathcal{O}_{n,\epsilon} \subset L_1(\mathbb{T})$ *such that*

$$\begin{aligned} &\mathcal{O} \cap \tilde{L}_1(\mathbb{T}) \neq \emptyset \quad \text{if } c = 0, \\ &\mathcal{O} \cap \left\{e = \bar{e} + \tilde{e} \in L_1(\mathbb{T}):\ -\epsilon < \bar{e} < 0\right\} \neq \emptyset \neq \mathcal{O} \\ &\quad \cap \left\{e = \bar{e} + \tilde{e} \in L_1(\mathbb{T}):\ 0 < \bar{e} < \epsilon\right\} \quad \text{if } c \neq 0, \end{aligned}$$

and for any $e \in \mathcal{O}$, *problem* (1) *has at least* $2n$ *geometrically different solutions.*

Proof. The case $c = 0$ follows directly from Corollary 4.4. Concerning the case $c > 0$, observe that it suffices to prove $\mathcal{O} \cap \{e = \bar{e} + \tilde{e} \in L_1(\mathbb{T}):\ 0 < \bar{e} < \epsilon\} \neq \emptyset$ $\forall \epsilon > 0$, since the remaining statements follow from the change of variables $\hat{u} = -u$, $\hat{g}(x) := -g(-x)$. In this way, this becomes a consequence of Corollary 4.4 and the fact that, as seen in Proposition 3.1, $\left\{\bar{e}_{c,T/m} - \frac{2\pi}{T} mc\right\}_m$ is a sequence of positive numbers converging to 0 as $m \to +\infty$. □

Proof of Theorem 1.1. In view of Corollary 4.5, we may assume there exist $n_0, m_0 \in \mathbb{N}$ such that $v_{c,T/m}$ is a trigonometric polynomial of degree not bigger that n_0 for all $m \geqslant m_0$. We choose $v_{c,T/m}$ so that $\|v_{c,T/m}\|_{L_\infty[0,T/m]} = 1$ and write

$$v_{c,T/m}(t) = \sum_{j=-n_0}^{n_0} \omega_{m,j} e^{\frac{2\pi m}{T} ijt}, \quad t \in \left[0, \frac{2\pi}{m}\right] \forall m \geqslant m_0,$$

where the complex coefficients $\{\omega_{m,j}\}_{-n_0 \leqslant j \leqslant n_0}$ verify $\omega_{m,-j} = \bar{\omega}_{m,j}$, $j = -n_0, \ldots, n_0$. The sequences $\{\omega_{m,j}\}_{m \geqslant m_0}$ are bounded for any $j = -n_0, \ldots, n_0$, and, after possibly passing to a subsequence, we may assume $\{\omega_{m,j}\} \to \omega_j$, $j = -n_0, \ldots, n_0$. Passing to the limit in the inequality $\sum_{j=-n_0}^{n_0} |\omega_{m,j}| \geqslant 1$ $\forall m \geqslant m_0$ we deduce that $\sum_{j=-n_0}^{n_0} |\omega_j| \geqslant 1$, and the trigonometric polynomial $v_c(t) := \sum_{j=-n_0}^{n_0} \omega_j e^{jit}$ is not the zero polynomial. We recall the differential equation verified by $v_{c,T/m}$,

$$v''_{c,T/m}(t) - cv'_{c,T/m}(t) + g'\big(u_{c,T/m}(t)\big)v_{c,T/m}(t) = 0, \quad 0 \leqslant t \leqslant \frac{T}{m},$$

or, what is the same,

$$v''_{c,T/m}\left(\frac{T}{2\pi m}t\right) - cv'_{c,T/m}\left(\frac{T}{2\pi m}t\right) + g'\left(u_{c,T/m}\left(\frac{T}{2\pi m}t\right)\right)v_{c,T/m}\left(\frac{T}{2\pi m}t\right) = 0,$$
$$0 \leqslant t \leqslant 2\pi.$$

Using the explicit form of $v_{c,T/m}$ as a trigonometric polynomial and passing to the limit as $m \to +\infty$, we deduce

$$v''_c(t) - cv'_c(t) + g'(t)v_c(t) = 0, \quad 0 \leqslant t \leqslant 2\pi,$$

since, as shown in Proposition 3.1, $u_{c,T/m}\big(\frac{T}{2\pi m}(t)\big) \to t$ uniformly with respect to $t \in \mathbb{R}$ as $m \to +\infty$. Here, we have an entire function which vanishes on a whole segment. It is, consequently, zero everywhere:

$$v''_c(z) - cv'_c(z) + g'(z)v_c(z) = 0 \quad \forall z \in \mathbb{C}. \tag{40}$$

Observe now that both nonzero trigonometric polynomials v_c and $v''_c - cv'_c$ have the same degree (recall the proof of Proposition 4.2 above). Therefore, by similar argument to those carried out in the proof of Theorem 5.2, they have the same number of roots, counting multiplicity. However, as given in (40), any root of v_c is a root of $v''_c - cv'_c$, so that they are in fact equal. It means $g'(z) = 1$ $\forall z \in \mathbb{C}$, which is a contradiction. The theorem is now proved. □

We finally have the following consequence of Lemma 2.3.

Theorem 4.6. *Assume $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$, let A_n, B_n, $n \geqslant 1$, be the sequences of Fourier coefficients of $v_{c,T}$ as defined in Proposition 4.2, and fix $k, k' \in \mathbb{R}$. If, for some $n_0 \in \mathbb{N}$, $A_{n_0}^2 + B_{n_0}^2 \neq 0$, then there exists an open set $\mathcal{O} \subset L_1([0,T])$ with $\mathcal{O} \cap \big\{e = \bar{e} + \tilde{e} \in L_1[0,T]\colon \bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c\big\} \neq \emptyset$, such that, for any $e \in \mathcal{O}$, problem (1) has exactly $2n_0$ geometrically different solutions.*

5. The conservative pendulum problem

Theorem 4.6 above can be criticized on the fact that it may not be easy to explicitly compute the Fourier series of the function $\nu_{c,T}$. In the conservative case, problem (28) is selfadjoint and things are simplified.

Corollary 5.1. *Let $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$, and let*

$$A_n := \frac{2}{T}\int_0^T u'_{0,T}(t)\cos\left(n\frac{2\pi}{T}t\right)dt,$$

$$B_n := \frac{2}{T}\int_0^T u'_{0,T}(t)\sin\left(n\frac{2\pi}{T}t\right)dt, \quad n \geqslant 1, \tag{41}$$

be the sequences of Fourier coefficients of $u'_{0,T}$. As before, fix $k, k' \in \mathbb{R}$. If, for some $n \in \mathbb{N}$, $A_n^2 + B_n^2 \neq 0$, then there exists an open set $\mathcal{O} \subset L_1([0,T])$ with $\mathcal{O} \cap \tilde{L}_1[0,T] \neq \emptyset$, such that for any $e \in \mathcal{O}$, problem (1) *has exactly* $2n$ *geometrically different solutions.*

In [9], it was seen that, in the special case of the conservative, pendulum equation (problem (3), $g(u) = \Lambda\sin(u)$), $u'_{0,T}$ cannot be a trigonometric polynomial, and this was used to see that the number of periodic solutions for the forced pendulum equation was not bounded as the forcing term varies in $C^\infty(\mathbb{T})$. In this work we have seen (Remark 3.2) that the analogous statement is not true for an arbitrary $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ function g. However, an improved argument can be used to prove that $u'_{0,T}$ is not a trigonometric polynomial when g belongs to an intermediate class of periodic nonlinearities, namely, those which are restriction to the real line of an entire function.

Theorem 5.2. *Assume that there exists an entire function whose restriction to the real line is g. Then, the number of $n \in \mathbb{N}$ such that*

$$\left|\int_0^T u'_{0,T}(t)e^{in\frac{2\pi}{T}t}\,dt\right|^2 \neq 0$$

is infinite. Consequently, there exists a sequence $\{n_m\}_{m\in\mathbb{N}} \to +\infty$ of natural numbers and, for each $m \in \mathbb{N}$, an open set $\mathcal{O}_{n_m} \subset L_1[0,T]$ with $\mathcal{O}_{n_m} \cap \tilde{L}_1[0,T] \neq \emptyset$, such that for any $\bar{e} \in \mathcal{O}_{n_m}$, problem (1) *has exactly* $2n_m$ *geometrically different solutions.*

Proof. To deny the statement of the theorem above is to say that $u'_{0,T}$ is a trigonometric polynomial. In complex notation, this can be written as

$$u'_{0,T}(t) = \sum_{j=-p}^{p} \omega_j e^{\frac{2\pi}{T}ijt}$$

for some complex coefficients $\{\omega_j\}_{j=-p}^{p}$, which should, furthermore, satisfy the relationship $\omega_{-j} = \bar{\omega}_j$. Of course, $u'_{0,T} \equiv cte$ is only possible if $g \equiv 0$, and thus, we should have $p \geqslant 1$, $\omega_p \neq 0$. On the other hand, the inequality $u'_{0,T}(t) > 0\ \forall t \in \mathbb{R}$ implies $\omega_0 > 0$. Now,

$$u_{0,T}(t) = \omega_0 t + \sum_{j=1}^{p} \frac{T}{2\pi j} i\left(\omega_{-j} e^{\frac{-2\pi}{T} ijt} - \omega_j e^{\frac{2\pi}{T} ijt}\right),$$

$$u'''_{0,T}(t) = -\sum_{j=-p}^{p} \left(\frac{2\pi j}{T}\right)^2 \omega_j e^{\frac{2\pi}{T} ijt},$$

and the equality $u'''_{0,T}(t) = g'(u_{0,T}(t))u'_{0,T}(t)$ becomes

$$\begin{aligned} &-\sum_{j=-p}^{p} \left(\frac{2\pi j}{T}\right)^2 \omega_j e^{\frac{2\pi}{T} ijt} \\ &\quad= g'\left(\omega_0 t + \sum_{j=1}^{p} \frac{T}{2\pi j} i\left(\omega_{-j} e^{\frac{-2\pi}{T} ijt} - \omega_j e^{\frac{2\pi}{T} ijt}\right)\right) \sum_{j=-p}^{p} \omega_j e^{\frac{2\pi}{T} ijt} \quad \forall t \in \mathbb{R}. \end{aligned}$$

Here, we have two entire functions which coincide on the real line. They are, consequently, equal on the whole complex plane

$$\begin{aligned} &-\sum_{j=-p}^{p} \left(\frac{2\pi j}{T}\right)^2 \omega_j e^{\frac{2\pi}{T} ijz} \\ &\quad= g'\left(\omega_0 z + \sum_{j=1}^{p} \frac{T}{2\pi j} i\left(\omega_{-j} e^{\frac{-2\pi}{T} ijz} - \omega_j e^{\frac{2\pi}{T} ijz}\right)\right) \sum_{j=-p}^{p} \omega_j e^{\frac{2\pi}{T} ijz} \quad \forall z \in \mathbb{C}. \end{aligned}$$

We multiply both sides of the equality above by $e^{i\frac{2\pi}{T} pz}$ to get

$$\begin{aligned} &-\sum_{j=0}^{2p} \left(\frac{2\pi(j-p)}{T}\right)^2 \omega_{j-p} e^{\frac{2\pi}{T} ijz} \\ &\quad= g'\left(\omega_0 z + \sum_{j=1}^{p} \frac{T}{2\pi j} i\left(\omega_{-j} e^{\frac{-2\pi}{T} ijz} - \omega_j e^{\frac{2\pi}{T} ijz}\right)\right) \sum_{j=0}^{2p} \omega_{j-p} e^{\frac{2\pi}{T} ijt} \\ &\quad \forall z \in \mathbb{C}. \end{aligned} \tag{42}$$

What is of interest for us in the equality above is the following: there exists an entire function $\vartheta : \mathbb{C} \to \mathbb{C}$ such that

$$-\sum_{j=0}^{2p} \left(\frac{2\pi(j-p)}{T}\right)^2 \omega_{j-p} e^{\frac{2\pi}{T} ijz} = \vartheta(z) \sum_{j=0}^{2p} \omega_{j-p} e^{\frac{2\pi}{T} ijt} \quad \forall z \in \mathbb{C}.$$

We consider the complex polynomials

$$q_1(z) := -\sum_{j=0}^{2p} \left(\frac{2\pi(j-p)}{T}\right)^2 \omega_{j-p} z^j, \qquad q_2(z) := \sum_{j=0}^{2p} \omega_{j-p} z^j.$$

Both of them have degree $2p$, so that both of them have $2p$ roots, counting multiplicity. Furthermore, 0 is not a root of either. However, the equality

$$q_1\big(e^{\frac{2\pi i}{T}z}\big) = \vartheta(z)q_2\big(e^{\frac{2\pi i}{T}z}\big) \quad \forall z \in \mathbb{C}$$

says that every root of q_2 is a root of q_1 with at least, the same multiplicity. We deduce that there exists $\varsigma \in \mathbb{C}$ such that $q_1 = \varsigma q_2$, that is

$$\vartheta(z) = \varsigma \quad \forall z \in \mathbb{C}.$$

In particular, $\vartheta(t) = g'(u_0(t)) = \varsigma$ $\forall t \in \mathbb{R}$. Thus, $\varsigma = 0$ and $g \equiv 0$, a contradiction. □

For pendulum-type equations without friction, a conservation of energy argument provides a explicit expression for $u_{0,T}$. Indeed, deriving the sum of kinetic plus potential energy along the trajectory $u_{0,T}$,

$$\mathcal{E}(t) = \frac{1}{2}u'_{0,T}(t)^2 + G\big(u_{0,T}(t)\big)$$

(here, G is any primitive of g), we find that the total energy does not change with time; there exists $\mathcal{E}_0 \in \mathbb{R}$ (total energy), such that

$$\mathcal{E}_0 = \frac{1}{2}u'_{0,T}(t)^2 + G\big(u_{0,T}(t)\big) \quad \forall t \in \mathbb{R}.$$

As $u'_{0,T}(t) > 0$ $\forall t \in \mathbb{R}$, we find that $\mathcal{E}_0 > \max_{\mathbb{R}} G$, and, further,

$$u'_{0,T}(t) = \sqrt{2\big(\mathcal{E}_0 - G\big(u_{0,T}(t)\big)\big)} \quad \forall t \in \mathbb{R}.$$

Equivalently,

$$\frac{u'_{0,T}(t)}{\sqrt{2(\mathcal{E}_0 - G(u_{0,T}(t)))}} = 1 \quad \forall t \in \mathbb{R}.$$

We consider the mapping

$$\mathcal{F}_{\mathcal{E}_0} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}}\, dy,$$

which is a increasing diffeomorphism in $\mathbb{R}$. Now,

$$\mathcal{F}_{\mathcal{E}_0}\big(u_{0,T}(t)\big) = t \quad \forall t \in \mathbb{R}$$

as it follows by simply deriving both sides of the equality. Therefore,

$$u_{0,T}(t) = \mathcal{F}_{\mathcal{E}_0}^{-1}(t) \quad \forall t \in \mathbb{R}. \tag{43}$$

In particular,

$$T = \mathcal{F}_{\mathcal{E}_0}(2\pi) = \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}}\, dy. \tag{44}$$

Previous result motivates the following question: Will it be possible, to find natural numbers n such that, with the notation of (41), $A_n^2 + B_n^2 = 0$? That is, may both terms of the same degree n in the Fourier series of $u'_{0,T}$ vanish simultaneously? If the answer were 'no,' at least for some 'nice' class of functions g, it would imply, as a consequence of Theorem 5.1, the existence, for each even number $2n$, of forcing terms $e \in L_1[0, 2\pi]$ such that (3) has *exactly* $2n$ solutions.

However, as seen in the introduction, this cannot be true in general, since, in case g is $\frac{2\pi}{p}$-periodic for some entire number $p \geqslant 2$, the number of geometrically different solutions to (1), if finite, is always an entire multiple of p. Indeed, what happens here is that the associated curve $u'_{0,T}$ is $\frac{2\pi}{p}$-periodic and consequently, all Fourier coefficients of degree not an integer multiple of p are zero.

On the other hand, numerical experiments carried out by the author seem to indicate that cosine Fourier coefficients of all orders

$$A_n := \int_0^T u'_{0,T}(t) \cos\left(n \frac{2\pi}{T} t\right) dt, \qquad n \geqslant 0,$$

are positive in the case of the pendulum equation $(g(u) = \Lambda \sin(u))$. However, we do not know a proof of this fact, and *the question remains open*.

We observe here that all sine Fourier coefficients of $u'_{0,T}$ vanish as soon as g is an odd function. Indeed, if this happens, the uniqueness of $u_{0,T}$ as a solution to (20) implies that

$$u_{0,T}(-t) = -u_{0,T}(t) \quad \forall t \in \mathbb{R},$$

and, consequently,

$$u'_{0,T}(-t) = u'_{0,T}(t) \quad \forall t \in \mathbb{R},$$

so that

$$B_n = \int_0^T u'_{0,T}(t) \sin\left(n \frac{2\pi}{T} t\right) dt = 0 \quad \forall n \in \mathbb{N}.$$

However, cosine Fourier coefficients can be shown to be positive when the time is big enough under our hypothesis (H). Indeed, it follows from (43) that

$$u'_{0,T}(t) = \frac{1}{\mathcal{F}'_{\mathcal{E}_0}(\mathcal{F}^{-1}_{\mathcal{E}_0}(t))} \quad \forall t \in \mathbb{R},$$

which implies

$$\begin{aligned} A_n &= \int_0^T u'_{0,T}(t) \cos\left(n \frac{2\pi}{T} t\right) dt = \int_0^T \frac{1}{\mathcal{F}'_{\mathcal{E}_0}(\mathcal{F}^{-1}_{\mathcal{E}_0}(t))} \cos\left(n \frac{2\pi}{T} t\right) dt \\ &= \int_0^{2\pi} \cos\left(n \frac{2\pi}{T} \mathcal{F}_{\mathcal{E}_0}(x)\right) dx = \int_0^{2\pi} \cos\left(n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}} dy\right) dx. \end{aligned} \tag{45}$$

Assume now that G attains its maximum only once on the interval $[0, 2\pi[$. Furthermore, assume the only point where this maximum is achieved is, precisely, π. Fix $n \in \mathbb{N}$ and let us make the time T diverge in expression (45). Simultaneously, $\mathcal{E}_0$, the energy of the trajectory, whose relation with T is given by (44), decreases to $\max_{\mathbb{R}} G$. Thus,

$$n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}}\, dy \to 2\pi n \chi_{]\pi, 2\pi]}(x), \quad 0 \leqslant x \leqslant 2\pi,$$

uniformly on compact subsets of $[0, \pi[\cup]\pi, 2\pi]$. Consequently,

$$A_n = \int_0^{2\pi} \cos\left(n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}}\, dy\right) dx \to 1 \quad \text{as } T \to +\infty.$$

We can use now Corollary 5.1 to prove Theorem 1.2.

Proof of Theorem 1.2. Of course, the maximum of G may not be attained precisely at π, but the number of solutions to problem (1) is not changed if g is translated on the real line, that is, replaced by $g(w + (\cdot))$, $w \in \mathbb{R}$. The theorem follows now from the discussion above. □

Proof of Theorem 1.3. We may well concentrate in the case $\Lambda > 0$, since the number of solutions of problem (3), is not changed as the periodic term $g(u)$ is replaced by $g(u + \pi)$. In this way, $G(u) = -\Lambda \cos(u)$ attains its maximum at π. Now, for any $0 \leqslant x < \frac{2\pi}{3}$ we have

$$\begin{aligned} 0 &\leqslant \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\, dy < \int_0^{2\pi/3} \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\, dy \\ &< \frac{1}{\sqrt{\Lambda}} \int_0^{2\pi/3} \frac{1}{\sqrt{1 + \cos(y)}}\, dy = \frac{2\sqrt{2}}{\sqrt{\Lambda}} \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \leqslant \frac{T}{3n\sqrt{2}} \end{aligned}$$

and, consequently,

$$\cos\left(n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\, dy\right) dx > \cos(\pi/3) = \frac{1}{2} \quad \forall x \in \left[0, \frac{2\pi}{3}\right[.$$

Therefore,

$$\begin{aligned} A_n &= \int_0^{2\pi} \cos\left(n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\, dy\right) dx \\ &= 2\int_0^{\pi} \cos\left(n \frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\, dy\right) dx \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{2\pi/3} \cos\left(n\frac{2\pi}{T\sqrt{2}}\int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\,dy\right)dx \\
&\quad + 2\int_{2\pi/3}^{\pi} \cos\left(n\frac{2\pi}{T\sqrt{2}}\int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}}\,dy\right)dx \\
&> \frac{2\pi}{3} - 2\frac{\pi}{3} = 0. \qquad \square
\end{aligned}
$$

Proof of Corollary 1.4. We simply observe that for any $g \in C^1(\mathbb{R}/2\pi\mathbb{Z})$, expression (45), which relates A_n, $\mathcal{E}_0$ and T, is analytic in these variables. Also, we know since (44) that $\mathcal{E}_0$ is an analytic function of T. An analytic function cannot be zero in an open set unless it is constantly zero, and, thus, Theorem 1.2 implies in fact Corollary 1.4. $\square$

Final Remarks. (1) Theorem 1.1 requires g to be the restriction to the real line of an entire function. A natural question is: What can be said when this hypothesis does not hold? This question was first partially answered by Katriel for oscillating functions $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$ with $g(\pi + x) = -g(x)\ \forall x \in \mathbb{R}$. In [5], he showed for problem (2) that, if, further, g is not a trigonometric polynomial, $\mathcal{S}_n \cap \tilde{L}_1(\mathbb{T}) \neq \emptyset$ with the notation from Theorem 1.1. The author has recently extended Katriel's results in [11], showing the first two hypothesis to be unnecessary for $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ and completing, in this way, Theorem 1.1.

(2) We chose an infinite-dimensional approach to problem (1). In order to prove Proposition 4.2 and Theorem 4.6—which are our key results giving rise to almost all others—it is also possible be to work instead with the Poincaré map, in just a two-dimensional setting, following the method and ideas in [9].

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